# MATHEMATICAL PROBABILITY IN ELECTION CHALLENGES

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Defeated candidates in primary elections sometimes challenge the results in court and collect evidence of irregularities in support of their claims that the contests should be rerun. Frequently, this evidence consists solely of proof that certain numbers of persons voted who were not qualified, with no evidence of fraud and no indication as to how such persons voted. How large must this group be before a new election should be ordered?

The New York Election Law provides that a new primary election may be ordered when the "irregularities . . . render impossible a determination as to who rightfully was . . . elected."[1] Consider a two-candidate contest in which the winner prevails by one hundred votes out of ten thousand. If there are 150 irregular voters, it is *possible* that more than 125 of them voted for the winner, so that their elimination would reverse the election. Does this possibility mean that the rightful winner cannot be determined within the meaning of the statute? The courts have answered this question with intuitive assessments of the probability that the result would be reversed if the challenged votes were removed. Thus, the Court of Appeals has articulated and applied the principle that the party attempting to impeach the results must show that the "irregularities are sufficiently large in number to establish the *probability* that the result would be changed by a shift in, or invalidation of, the questioned votes."[2]

Two polar assessments of the relevant probabilities may be illustrated by comparing *Ippolito v. Power*[3] with *De Martini v. Power*.[4] In *Ippolito*, the winner's plurality was 17 votes out of 2,827; there were 101 suspect or invalid votes. The court affirmed the lower court's ordering of a new election. Evidently relying on intuition, the court concluded that "it does not strain the probabilities to assume a likelihood that the questioned votes produced or could produce a change in the result."[5]

In *De Martini*, out of 5,250 votes, 136 were declared irregular and in-

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1. N.Y. Election Law § 330(2) (McKinney 1964). There is no comparable statutory provision for ordering a new general election. For a comprehensive discussion of section 330 challenges, see Note, *Primary Challenges in New York: Caselaw Coleslaw v. Election Protection*, 73 Colum. L. Rev. 318 (1973).

2. Ippolito v. Power, 22 N.Y.2d 594, 597-98, 241 N.E.2d 232, 233, 294 N.Y.S.2d 209, 211 (1968) (emphasis added). This standard was quoted with approval and applied in De Martini v. Power, 27 N.Y.2d 149, 151, 262 N.E.2d 857, 858, 314 N.Y.S.2d 609, 610 (1970).

3. 22 N.Y.2d 594, 241 N.E.2d 232, 294 N.Y.S.2d 209 (1968).

4. 27 N.Y.2d 149, 262 N.E.2d 857, 314 N.Y.S.2d 609 (1970).

5. 22 N.Y.2d at 598, 241 N.E.2d at 233, 294 N.Y.S.2d at 211.



validated, no fraud being involved. The winner's plurality was 62 votes. The lower courts and Court of Appeals differed in their estimates of the relevant probability. The Supreme Court ordered a new election because "it is not beyond likelihood that the small difference of 62 votes could be altered in a new election."[6] The Appellate Division unanimously affirmed. In reversing, the Court of Appeals observed that the majority of the winner would not evaporate unless at least 99 votes—*i.e.*, at least 72.8% of the irregularities—had been cast in her favor. The court found this unlikely: "It taxes credulity to assume that, in so close a contest, such an extreme percentage of invalid votes would be cast in one direction." It concluded that "a valid determination is not rendered impossible . . . by the remote possibility of a changed result . . . ."[7]

Subjective estimates of the relevant probabilities have thus varied. There is, however, no reason to leave matters on a purely subjective basis. Using an assumption about the character of invalid voting which will be defensible in many cases, the relevant probabilities can be readily computed.

Consider all the votes cast in a primary election as balls placed in an urn: black balls which predominate are those votes cast for the winner; white balls are for the loser. A certain number of balls representing the irregular voters are then withdrawn at random from the urn, an operation which corresponds to their invalidation. What is the probability that, after the withdrawal, the number of black balls no longer exceeds the number of white? Note the key assumption that the balls are withdrawn at random, *i.e.*, that each ball has the same probability of being withdrawn. In terms of the real election situation, each voter is deemed to have the same probability of casting an invalid vote. This assumption will of course be untenable if evidence of fraud or patterns of irregular voting indicates that a disproportionate number of improper votes were cast for one candidate. But in the absence of such evidence, the assumption of random distribution of the improper votes is warranted. Whether or not mathematics is used to assess the probabilities, some implicit or explicit view as to the pattern of irregular voting seems inevitable. The assumption that each voter had an equal probability of casting an improper vote is the only neutral and non-arbitrary view that can be taken when there is no evidence to indicate that the probabilities are not equal. Thus in *Ippolito*, *De Martini*, and other cases,[8] where there was no evidence to disturb the assumption of randomness, the mathematical probability analysis depicted by the urn model is a correct expression of the intuitive probability used by the Court of Appeals in formulating the burden of proof standard for a new election.

6. Quoted at 27 N.Y.2d at 151, 262 N.E.2d at 857, 314 N.Y.S.2d at 610. Despite the court's casual language, the proper inquiry is whether the election result at bar was affected through irregular voting, not whether a new election would yield a different result.

7. 27 N.Y.2d at 151, 262 N.E.2d at 858, 314 N.Y.S.2d at 611.

8. *E.g.*, Posner v. Power, 18 N.Y.2d 703, 220 N.E.2d 269, 273 N.Y.S.2d 480 (1966); Santucci v. Power, 25 N.Y.2d 897, 252 N.E.2d 128, 304 N.Y.S.2d 593 (1969).

In terms of the urn model, the mathematical probability of a reversal is simply the number of combinations in which the balls representing the invalid votes may be withdrawn from the urn so as to produce a reversal, divided by the total number of combinations in which these balls may be withdrawn from the urn. If the number of votes in question was small enough, these combinations could be simply enumerated. In most practical applications, however, the combinations are far too numerous for simple counting. An approximation of this probability with sufficient accuracy for legal purposes can be obtained by using the formula

$$Z = d\sqrt{\frac{s-k}{sk}}$$

where $d$ is the winner's plurality; $s$ is the number of votes cast either for the winner or his challenger; and $k$ is the number of invalid votes cast either for the winner or his challenger.[9] The value of $z$ determines the probability of reversal; as it increases this probability declines rapidly. The following table gives some benchmarks:[10]

| *Value of z* | *Probability of Reversal* |
|---|---|
| 0.5 | .81 |
| 1.0 | .16 |
| 1.5 | .07 |
| 2.0 | .02 |
| 3.0 | .001 |

The application of this formula demonstrates a first good reason for using mathematical methods, namely, that uneducated intuition is not a reliable guide to the true probabilities. On the facts in *Ippolito*, analysis indicates about a 5 percent chance that the election would have been reversed by the removal of the irregular votes.[11] Did the court realize that the chance was this small when

9. The derivation of this formula appears in the Appendix. The formula uses the total vote for the election, and this in general should be sufficiently accurate. If, however, votes for subdistricts are available, and if the pattern of voting varied substantially from subdistrict to subdistrict, it may be desirable to make separate computations for each subdistrict and to aggregate them as shown in the subdivision formula in the Appendix.

If there are only two candidates and no other contest, $k$ will equal the number of challenged votes. If there are more than two candidates or if there are other contests (*e.g.*, a primary for another party being run simultaneously, as is frequently the case), some of the invalid votes may have been cast for the third candidate or in the other contest. To account for this effect precisely would raise mathematical complexities. Frequently, it can be demonstrated that the probability of a reversal is extremely small even if it is assumed that all the invalid votes were cast for the winner or his challenger, an assumption which clearly generates a larger probability of reversal than if a more realistic assumption were adopted. In closer cases, the probabilities fairly may be approximated by assuming that the winner and challenger combined received the same proportion of invalid votes as they did of the total vote for all candidates and contests. On this assumption $k$ will be proportionately smaller than the total number of invalid votes.

10. $z$ is simply a standard normal variate. A full table of values and associated probabilities may be found in most textbooks on statistics. *See, e.g.*, FREUND, MATHEMATICAL STATISTICS, Table III at 366 (1962).

11. $z = 17\sqrt{\frac{2827-101}{(2827)(101)}} = 1.6$. This is a value associated with a 5 percent probability of reversal.



it concluded there was a "likelihood" of reversal?[12] One cannot be sure, but 5 percent seems a rather small probability to be termed a likelihood. If so, *Ippolito* may well be wrong in the sense that the court was acting on an overestimate of the probability of reversal.[13]

In *De Martini*, although the Appellate Division found a reversal "not beyond likelihood," the mathematical test demonstrates the contrary. The chance that the winner's plurality of only 62 votes would be eroded to a tie or loss by the removal of 136 irregular votes was less than one in a million.[14] Thus the Court of Appeals, reversing, was clearly correct when it concluded that the chance of a reversal was "remote." The court's judgment on this question may have been influenced by a statistical analysis in appellant's brief which, using a weak form of mathematical estimation, demonstrated that the probability of a reversal was less than 3.7 percent.[15] If the court had this figure in mind when it characterized the probability of removal as "remote," it evidently made a mistake in *Ippolito* when it found that this probability was substantial although the true figure was only 5 percent.

The court was even more clearly mistaken in *Santucci v. Power*,[16] when it affirmed an order directing a new election on the basis of 640 irregularities and a winner's plurality of 95 votes. In citing *Ippolito*, the court evidently believed that the probability of a reversal was substantial. Mathematical analysis demonstrates, however, that this probability was in fact less than one in ten thousand.[17]

In assessing the probability of reversal required to order a new election,

12. No mathematical analysis was presented in the briefs for either party.

13. In *Ippolito* the court relied on its affirmance of a new election in Nodar v. Power, 18 N.Y.2d 697, 220 N.E.2d 267, 273 N.Y.S.2d 273 (1966), which it said involved "almost identical facts." But the plurality in *Nodar* was 27 votes, more than 50 percent larger than in *Ippolito*. Although the numbers in both cases were small, the difference is of some consequence. The formula applied to the *Nodar* facts is $z = 27\sqrt{\frac{1417-109}{(1417)(109)}} =$ 2.4, a value of $z$ associated with a 1 percent probability of reversal. This is five times smaller than in *Ippolito* and surely should be deemed inconsequential for legal purposes.

14. $z = 62\sqrt{\frac{5250-136}{(5250)(136)}} = 5.2$, a value of $z$ associated with a probability of less than one in a million.

15. Brief for Appellant, Appendix. The method used was Chebychev's inequality.

16. 25 N.Y.2d 897, 252 N.E.2d 128, 304 N.Y.S.2d 593 (1969).

17. After a proportionate reduction in the number of irregularities to account for their distribution among candidates other than the first two (in accordance with the procedure stated in note 9 *supra*) the numbers in *Santucci* were:

$$z = 95\sqrt{\frac{116{,}057-448}{(116{,}057)(448)}} = 3.8.$$

On the other hand, a defensible decision was made in Posner v. Power, 18 N.Y.2d 703, 220 N.E.2d 269, 273 N.Y.S.2d 480 (1966) where the court found 370 to 412 irregularities in a four-candidate contest in which the plurality of the winner was 24 votes. Assuming 412 irregular votes and allocating a proportionate number of irregularities to the two last-place candidates, the probability of a reversal was 9 percent, which would seem sufficient to justify the court's affirmance of the order directing a new election.

it should be recalled that this probability will be less than 50 percent regardless of the number of invalid votes removed. Consequently, the critical probability must be some not insubstantial figure, but short of the 50+ percent implied by a "more-likely-than-not" test.[18]

A second reason for using a mathematical approach is to prevent resort to misleadingly simple rules of thumb. As the formula shows, the probability of reversal depends principally upon the size of the plurality and the number of irregularities (and less sensitively on the total vote). Its results cannot, however, be expressed as a simple relation between these variables. For example, if the plurality is small enough, a substantial probability of reversal will exist if the number of irregularities is twice the plurality; but if the plurality is large, irregularities four or five times larger may be insufficient to cast even a shadow of doubt on the results.[19] Thus, the rule used in some cases that a new election will be ordered when the number of irregularities exceeds a certain multiple of the plurality is incorrect as a method of intuitive estimation and misleading as a method of analyzing the precedents.[20]

A third reason for using mathematical techniques is that uncertainty over intuitive estimates of probability tends to obscure significant legal issues which arise in certain cases. An examination of the recent case of *Lowenstein v. Larkin*, in which the Appellate Division unanimously set aside Congressman John Rooney's primary victory over Allard Lowenstein,[21] and was affirmed on the opinion below by the Court of Appeals,[22] serves to illustrate this point.

In ordering a new election, the Appellate Division purported to apply the rule of section 330 that the election was "characterized by such . . . irregularities as to render impossible a determination as to who rightfully was nominated." If the court meant by this that the invalid votes made the result uncertain, a judicial recall of the election was not supported by the facts recited in the court's opinion.

The court found that, among other instances of misconduct, due to errors

18. The Court of Appeals, probably unwittingly, has sometimes sounded as if it might apply such a preponderance test. *See, e.g.*, Ippolito v. Power, 22 N.Y.2d 594, 597-98, 241 N.E.2d 232, 233, 294 N.Y.S.2d 209, 211 (1968) (the irregularities "would not be sufficiently large in number to establish *the probability* that the result would be changed by a shift in, or invalidation of, the questioned votes.") (emphasis supplied).

19. A plurality of five votes with twice as many irregularities might justify a new election since the probability of reversal could be approximately 5 percent. On the other hand, with a plurality of 100 votes, even four times as many irregularities would not justify a new election since the probability of reversal would be less than one in a million.

20. *See, e.g.*, Santucci v. Power, 25 N.Y.2d 897, 252 N.E.2d 128, 304 N.Y.S.2d 593 (1969) (order directing a new election affirmed when irregularities were analyzed as being "at least six and one-half times the winning margin"); Posner v. Power, 18 N.Y.2d 703, 220 N.E.2d 269, 273 N.Y.2d 480 (1966) (order directing a new election affirmed when irregularities were analyzed as being "15 to 17 times the margin of winning votes"); Merola v. Power, 60 Misc. 2d 245, 248, 303 N.Y.S.2d 229, 232 (Sup. Ct.), *aff'd mem.*, 33 App. Div. 2d 514 (1st Dep't 1969) (new election ordered and prior cases analyzed on the basis that the irregularities exceeded three times the plurality).

21. 40 App. Div. 2d 604, 335 N.Y.S.2d 799 (2d Dep't 1972) (*mem.*).

22. 31 N.Y.2d 654, 288 N.E.2d 133, 336 N.Y.S.2d 249 (1972) (*mem.*).



by the Board of Elections, "hundreds of persons" were turned away from the polling places to which they had been assigned and that others were improperly permitted to vote because challenges were ignored or because their registrations should have been cancelled, but were not. The exact impact of these mistakes evidently was unknown, the court being able to find only that "at least" 1,920 irregular votes were cast in the election out of a total of 29,567.

The case was removed from the ordinary run by evidence of official favoritism for Rooney. Campaigning for Rooney at polling places and ignoring the challenges of Lowenstein poll watchers were among the practices the court found to have tainted the election. There was, however, nothing in the circumstances cited in the court's opinion to suggest that those improperly permitted to vote favored Rooney while those improperly excluded or inhibited from voting favored Lowenstein. On the court's findings, one could assume no more against Rooney than that the improper voters (and those inhibited or discouraged from voting) were a random sample of the total voting population.

The number of irregularities was large—at least 1,920 votes—and the court may have believed that this was sufficient to create a substantial probability of reversal, even on a random sample basis. Rooney's plurality, however, was also large—890 votes—and the number of irregular votes required to generate even a small probability of reversal rises very rapidly with the winner's plurality, probably much more rapidly than most statistically uneducated persons would suppose. In fact, given this large plurality, it would have taken more than 27,000 irregularities to create even a 2 percent probability of a reversal. Lowenstein did not claim irregularities on anything approaching this scale.[23]

The mathematical demonstration illuminates the nature of the specific findings of fact or legal conclusions that would have been required to justify a new election. If the court relied on the traditional standard—an appreciable probability of reversal—it would have had to conclude that the mistakes of the Board or its officials were not in fact neutral, but favored Rooney, in the sense that those voting improperly favored Rooney while those improperly inhibited from voting favored Lowenstein. No such findings appear in the opinion, although the evidence might have justified this conclusion.

Alternatively, the court could have taken an enlarged view of section 330, and held that serious irregularities in procedure by Board officials, particularly deviations favoring one candidate, constituted sufficient grounds for ordering a new election, even in cases in which it could not be shown that

23. In Celler v. Larkin, 71 Misc. 2d 17, 335 N.Y.S.2d 791 (Sup. Ct.), *aff'd mem.*, 40 App. Div. 2d 603, 335 N.Y.S.2d 801 (2d Dep't), *aff'd mem.*, 31 N.Y.2d 658, 288 N.E.2d 135, 336 N.Y.S.2d 251 (1972), Elizabeth Holtzman won the Democratic primary for Congress by some 600 votes out of 30,000. In the trial of Celler's challenge to her victory, Professor Robbins testified that Celler would have had to show more than 16,900 irregularities out of the 30,000 votes cast to create even a one-in-a-thousand chance of a reversal. (T. at 422-23). The courts rejected the challenge.

such errors affected, or were likely to have affected, the outcome. Since section 330 does not seem to read this broadly, and since a liberal reading should be "confined to the subject matters plainly enumerated therein,"[24] an extension of section 330 to cover such cases would be a significant development in the law meriting judicial discussion.[25]

Nothing of that sort appears in the opinion of the Appellate Division. Instead, its decision appears to rest on an undifferentiated mix of disapproval for official irregularities, and hazy or mistaken intuitive notions of probability. If the method of mathematical probability had been recognized in *Lowenstein*, the attention of the courts would have been focused on the legal issues which were critical in that case and which are likely to arise in future cases: whether there was sufficient evidence to reject the assumption of random distribution of the improper votes, and if not, whether official irregularities favoring one candidate, but without demonstrable effect on the result, are a permissible statutory ground for ordering a new election.

## APPENDIX

*Global analysis of challenged elections.*

Suppose the winning candidate $A$ has $a$ votes and the losing candidate $B$ has $b$ votes, with $a$ greater than $b$. Of the total of $s = a + b$ votes, $k$ are removed at random. Let $x$ denote the number of these $k$ votes which are for $A$. There will be a *reversal* if after the removal there are at least as many votes for $B$ as for $A$; *i.e.*, if $a - x \leqq b - (k - x)$, or

$$x \geqq (a - b + k)/2. \qquad (1)$$

In a random withdrawal of $k$ balls from $s = a + b$ balls in an urn, the number $x$ of $A$ balls withdrawn is a random variable with a "hypergeometric" distribution (sampling without replacement from a finite population); the mean and variance of $x$ are given by

$$\Sigma(x) = ka/s, \qquad \mathrm{Var}(x) = kab(s-k)/s^2(s-1).$$

The standardized random variable

$$z = (x - \Sigma(x))/\sqrt{\mathrm{Var}(x)}$$

is then (replacing $s - 1$ by $s$ for simplicity) given by the formula

$$z = \frac{sx - ka}{\sqrt{\left(kab\left(1 - \frac{k}{s}\right)\right)}};$$

$z$ has mean 0 and variance 1. The condition (1) for reversal in terms of $z$ is

24. Matter of Hyer, 187 Misc. 946, 948, 63 N.Y.S.2d 874, 876 (Sup. Ct. 1946).

25. Perhaps the necessary interpretation could be made by leaning heavily on the word "rightfully" in section 330. Arguably, the winner cannot claim to be rightfully selected in an election marred by serious breaches of the rules in his favor by officials charged with the duty of neutrality.



that the numerator of $z$ be greater than or equal to $s(a - b + k)/2 - ka = (s - k)(b - a)/2$; *i.e.* that $z$ be greater than or equal to the constant

$$C = \frac{(b-a) \cdot \sqrt{(s(s-k))}}{2\sqrt{(kab)}}.$$

Now, since $a + b = s$, $ab \leqq s^2/4$, with approximate equality when $a$ and $b$ are nearly equal. In any case, we see that

$$C \geqq (b - a) \cdot \sqrt{\left(\frac{1}{k} - \frac{1}{s}\right)}. \qquad (2)$$

Since $z$ is approximately normally distributed, *the probability of a reversal is about equal to the probability that a standard normal random variable $z$ will exceed the constant given by the right-hand side of* (2).

*Analysis by subdistricts.*

If $k_i$ votes are removed in the $i$th district, an elaboration of the global analysis shows that the condition for a reversal is that an approximately standard normal random variable $z$ will exceed the constant

$$\frac{(a - b + k) - 2\left[\text{ sum over } i \text{ of } \left(\frac{k_i a_i}{a_i + b_i}\right)\right]}{2 \cdot \text{sq. rt.} \left[\text{ sum over } i \text{ of } \left(\frac{k_i a_i b_i}{a_i + b_i}\right)\right]}$$

where again $k =$ [sum over $i$ of $k_i$] $=$ total number of votes removed from the $a + b$ votes.